QUADRATIC MODELING – PART II

We continue our discussion of the uses of quadratic functions (parabolas) to solve business problems involving both revenue and expenses. We'll also solve problems involving the maximization of the area of a rectangular region using a given amount of fencing.

EXAMPLE 1: **The revenue function at Widget, Inc., where w is the number of widgets produced, is given by $R(w) = 2w^2 + 100w - 6$, while the expense function is $E(w) = 7w^2 - 50w + 1$. Find the production level of widgets that will yield a maximum profit, and then calculate the profit.**

Solution: Revenue is the money received from selling a product or service, and expenses are the costs incurred in providing that product or service to the customer. Profit is the difference between revenue and expenses, so we can write the formula

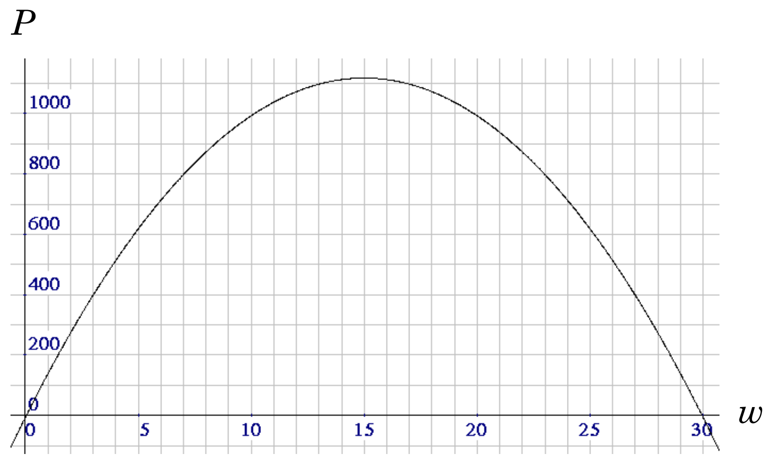
$$P(w) = R(w) - E(w)$$

$$\Rightarrow \quad P(w) = (2w^2 + 100w - 6) - (7w^2 - 50w + 1)$$

$$\Rightarrow \quad P(w) = 2w^2 + 100w - 6 - 7w^2 + 50w - 1$$

$$\Rightarrow \quad P(w) = -5w^2 + 150w - 7$$

This formula shows that the profit, P, is a quadratic function of the number of widgets, w. So its graph is a parabola opening down (since $-5 < 0$). Because we're trying to determine the maximum profit, we observe that the maximum value of P will be at the vertex (top) of the parabola. A rough sketch of the profit function (the parabola) should make this concept clear.



The w-coordinate of the vertex is given by the formula $\frac{-b}{2a}$, where $a = -5$ and $b = 150$. So, $w = \frac{-b}{2a} = \frac{-150}{2(-5)} = \frac{-150}{-10} = 15$.

Hence, the production level of widgets should be 15. This maximizes the profit. Fewer than 15 widgets would result in less profit; more than 15 widgets would also reduce profit. The optimum number of widgets is 15.

Now to calculate the profit on the 15 widgets:

$$P(w) = -5w^2 + 150w - 7$$

$$\Rightarrow \quad P(\mathbf{15}) = -5(\mathbf{15})^2 + 150(\mathbf{15}) - 7$$

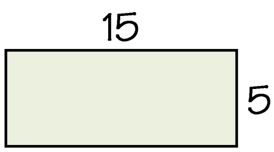
$$\Rightarrow \quad P(15) = 1118$$

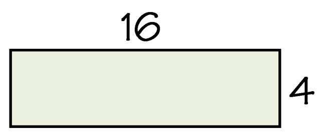
Producing **15** widgets gives a maximum profit of **$1,118**

This result is consistent with the graph of the parabola.

<u>EXAMPLE 2:</u> **Farmer Gertrude has 40 feet of fence to build a rectangular pigpen. What should the dimensions of the pigpen be so that it will enclose the <u>maximum area</u>? What will the area be with these dimensions?**

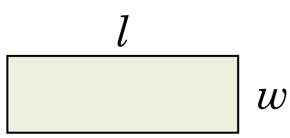
<u>Solution:</u> As a student, I remember thinking that the given 40 feet of fence would produce the same area regardless of the dimensions of the rectangle. However, consider the following rectangles:





Each rectangle has the same perimeter (the 40 feet of fence), but the areas (75 sq ft and 64 sq ft) are different — the first rectangle contains 11 additional sq ft, perhaps room for another pig.

The solution begins by labeling a rectangular pigpen:



Now, the **perimeter** of the rectangle is given to be 40 feet. Thus,

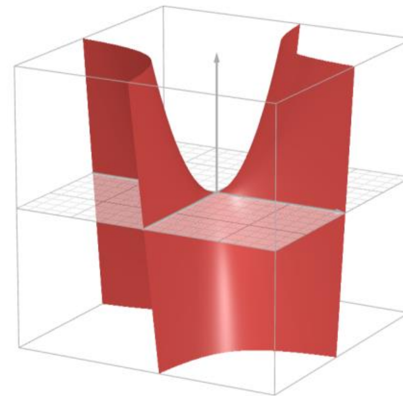
$$2l + 2w = 40$$

Also, the **area** of the rectangle is to be made as large as possible.

$$A = lw$$

> The perimeter equation is called the **constraint** because it limits the possible dimensions of the rectangle.
>
> The area formula is called the **objective function** because it represents the quantity we wish to maximize.

We'd like to graph this area function, $A = lw$, but the problem with this formula is that the area is a function of too many variables (we can't graph three variables in a two-dimensional coordinate system). But we can tie the perimeter formula to the area formula with a little substitution, thereby expressing the area, A, as a function of one variable instead of two:



This is the graph of $A = lw$, not helpful until Calculus.

Solve for l in the perimeter formula:

$$2l + 2w = 40 \Rightarrow 2l = 40 - 2w$$

$$\Rightarrow \quad l = \frac{40 - 2w}{2} \Rightarrow l = \frac{40}{2} - \frac{2w}{2} \Rightarrow \underline{l = 20 - w}$$

Now substitute $20 - w$ for l in the area formula:

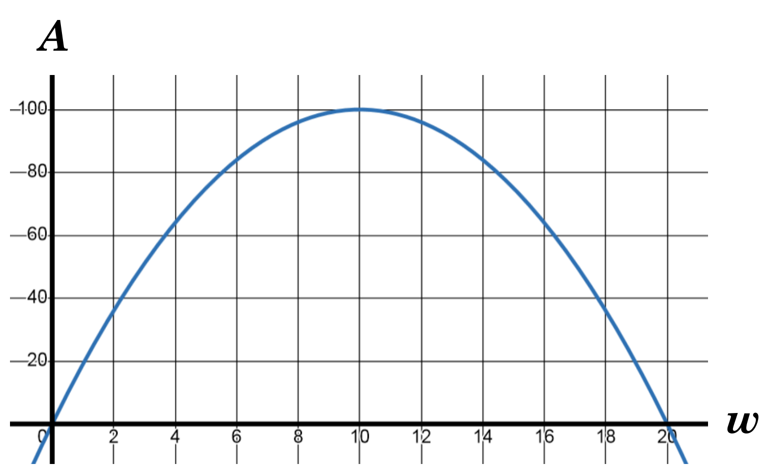
$A = lw$ (the area formula)

$\Rightarrow A = (\underline{20 - w})w$ (letting $l = 20 - w$)

$\Rightarrow A = 20w - w^2$ (distribute)

$\Rightarrow A = -w^2 + 20w$ (better form)

We have written the area as a function of just one variable, w. We want to find the value of w that will maximize A. We can find this value of w because the equation we just wrote is a parabola:



Looking at the graph, we conjecture that the vertex — the maximum point on the curve — occurs at the point (10, 100). This is verified by using the formula for the vertex, where the w-coordinate is given by

$$w = \frac{-b}{2a} = \frac{-20}{2(-1)} = \frac{-20}{-2} = 10$$

Thus, the width is 10 feet. Using the formula $l = 20 - w$ we see that the length is also 10 feet. The rectangle of maximum area is therefore actually a

$$\boxed{10 \text{ ft} \times 10 \text{ ft square}}$$

whose area is 100 sq ft.

Homework

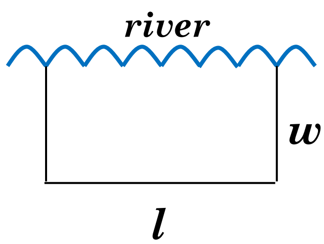
1. The revenue function is $R(x) = 2x^2 - 5x - 25$, while the expense function is $E(x) = 7x^2 - 85x + 75$. Find the production level that will yield a maximum profit. What is the profit?
   Recall: Profit = Revenue – Expenses.

2. The revenue function is $R(x) = 2x^2 + 2000x + 200$, while the expense function is $E(x) = 11x^2 + 596x - 50$. Find the production level that will yield a maximum profit. What is the profit?

3. To protect his corn from Farmer Gertrude's pigs (Gertrude made the 10 ft × 10 ft pigpen alright, but she made it only 9 inches tall!), Uncle Eb needs to enclose his corn in a rectangular region, using only 84 feet of fence. He would like as much area as he can get with his fence. What should be the dimensions of Uncle Eb's cornfield?

4. Same as the previous problem, but assume 180 feet of fence.

5. Rancher Millie liked what Gertrude and Uncle Eb had done, so when she needed to build a cow corral, she consulted them about her problem. Gertrude and Eb told Millie that a square would definitely be the best shape, since a square gave both of them the most area for the given amount of fence. What Millie didn't think was relevant to tell her friends was that a river would be used as one side of the corral; that is, only three sides of the rectangle would have to be fenced:



If Rancher Millie has 1000 feet of fence to build the 3-sided rectangle, what dimensions would yield her the maximum area?

6. The same scenario as the previous problem, except Rancher Millie has 1200 feet of fence.

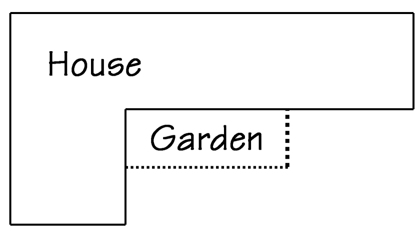
Review Problems

7. The revenue function for the widget factory is $R(w) = 23w^2 + 950w + 95$, and the expense function is $E(w) = 33w^2 + 350w - 5$. How many widgets should be produced to achieve the maximum profit?

8. What dimensions should a rectangular region be if the perimeter must be 100 ft and if the maximum area is desired?

9. A rectangular corral is to be built with 144 feet of fence, using the side of a barn as one of the sides of the rectangle. If the goal is to enclose the maximum number of square feet, what should the dimensions of the rectangle be?

10. Bruce has 60 feet of fence with which to enclose a rectangular region for his flower garden. But an inside corner of the yard will provide two of the four sides of the rectangle. Find the dimensions of the rectangle that will provide Bruce with the maximum possible area for his garden.



Solutions

1. 8 items; \$220

2. 78 items; \$55,006

3. 21 ft × 21 ft

4. 45 ft × 45 ft

5. Hint: It's not a square

6. 600 ft × 300 ft

7. 30 widgets

8. 25 ft × 25 ft

9. 72 ft × 36 ft

10. 30 ft × 30 ft

Education is not

a preparation for life;

education is life itself.

John Dewey